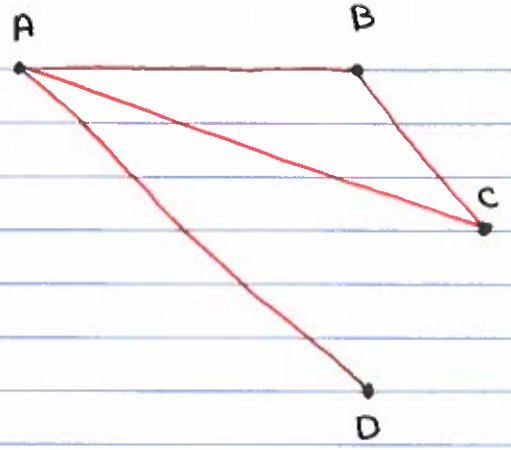
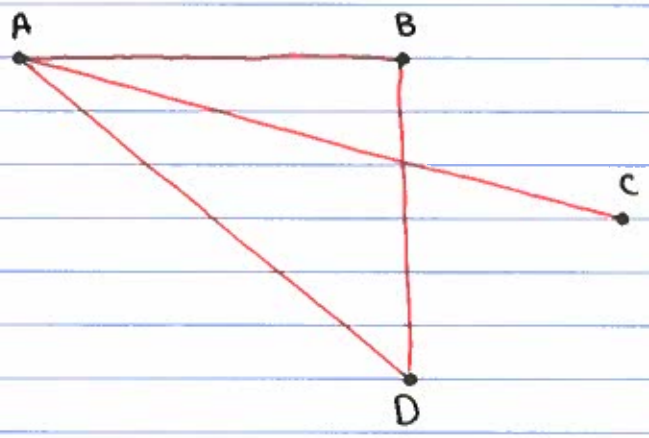
Ramsey Theory Example

Consider any 6 points on the plane and color all edges which connect these 6 points either red or blue. Show that there must always be a red triangle or blue triangle.

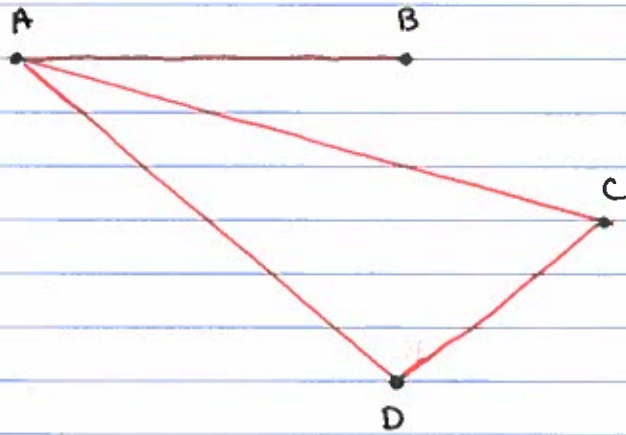
Soln:
Consider one of these points, call it A. Then, A is connected to five more points and so there are five edges which terminate at A. Each of these five edges is either red or blue. We have 5 edges (pigeons) and 2 colors (pigeonholes) and so by P.P, there must be at least 3 edges which terminate at A which are either red or blue. Assume there are 3 edges that are red. Let B, C, D be the endpoints of these 3 red edges. I.e. We have the red segments AB, AC, AD. If either of the segments BC, BD, CD is red, then we have a red triangle. If none of the edges BC, BD, CD is red, then they are all blue and hence triangle BCD is blue. Hence, in all cases, there must exist a red or blue triangle.



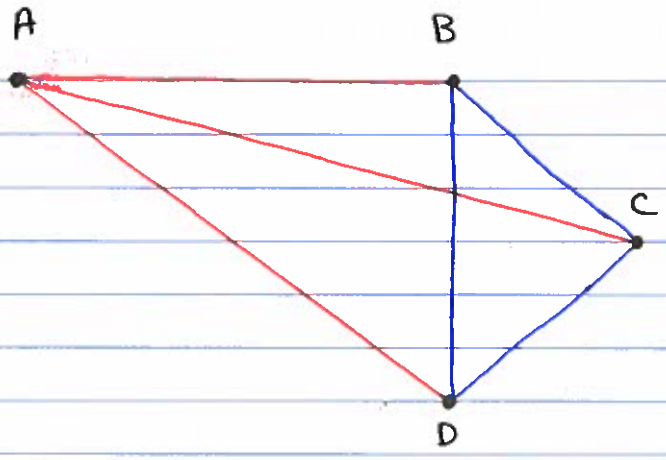
If BC has a red edge, then we have a red triangle.



If BD has a red edge, then we have a red triangle.



If CD has a red edge, then we have a red triangle.



If none of BC, BD and CD are red, then they must be blue and we have a blue triangle.